# Symbolic dynamics of NMR-laser chaos

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For the extended Bloch-type model of the NMR laser a binary partition is determined from tangencies between forward and backward foliations. It is found that both forward and backward symbolic sequences are ordered as those of the Hénon map with a positive Jacobian. From a finite set of tangencies, new admissible periods are obtained, among which some were regarded as forbidden. A method to construct allowed chaotic sequences is also given.

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Symbolic dynamics of one-dimensional maps on an interval is well understood [1,2]. For the simplest case of the unimodal map, a binary generating partition may be introduced by splitting the interval at the critical point. An orbit can then be encoded with a symbolic sequence by assigning either letter $R$ or $L$ to an orbit point, depending on whether it falls to the right or left side of the critical point, respectively. The kneading sequence, which is the forward sequence of the critical value, determines all the admissible sequences of the map. According to the kneading theory, all sequences are well ordered. The kneading sequence is the greatest. No allowed sequence would contain any shifted subsequence greater than the kneading sequence.

The extension of the symbolic dynamics of the unimodal map to that of the two-dimensional Hénon map [3] $(x,y)\to(1-ax^2+by,x)$ is by no means trivial. The first problem is to construct a "good" binary partition for this system. In Ref. [4], by considering all "primary" homoclinic tangencies, a method was proposed. Once the binary partition is practically determined, each point, or its orbit, may be associated with a doubly infinite symbolic sequence [5] $S=\ldots s_{\bar 2}s_{\bar 1}\bullet s_0s_1\ldots$, where $s_0$ indicates the code of the point. The forward sequence $\bullet s_0s_1\ldots$ and backward sequence $\ldots s_{\bar 2}s_{\bar 1}\bullet$ correspond to its forward and backward orbits, respectively. In Ref. [5] the ordering rules for forward and backward sequences were discussed, and the symbolic plane for a metric representation of the ordering was constructed. It was pointed out that every primary homoclinic tangency cuts out a rectangle of forbidden sequences in the symbolic plane. By generalizing stable and unstable manifolds to forward and backward foliations [6,7], homoclinic tangencies can be generalized to tangencies between the two classes of foliations [8].

Symbolic dynamics of maps can be applied to the study of differential equations by considering the Poincaré map in a surface of intersection. In Ref. [9] kneading sequences of the unimodal map were identified to some stable periodic orbits of the forced Brusselator. The Lorenz model was discovered to be related to the symbolic dynamics of the antisymmetric cubic map [10]. In these two examples only the symbolic description of one-dimensional (1D) maps was experimentally used for stable periodic orbits. Recently, in Ref. [11], a two-dimensional (2D) symbolic description was proposed for the model of NMR-laser chaos, the following extended Bloch-type laser (EBL) model:

$$\begin{aligned}\dot x &= \sigma[y-x/(1+A\cos\omega t)] ,\\ \dot y &= -y(1+ay)+rx-xz ,\\ \dot z &= -bz+xy .\end{aligned} \tag{1}$$

The meaning of the variables $x,y,z$, and the parameters $\sigma$, $A$, $\omega$, $a$, $r$, and $b$ can be found in their papers. A binary generating partition was approximately obtained from some unstable orbits up to period 9. However, the ordering rules of forward and backward sequences as well as the admissibility conditions for allowed sequences have not been touched. In this Brief Report we present a full analysis of symbolic dynamics for the EBL model.

The extension of symbolic dynamics of maps from 1D to 2D is made by decomposing a 2D map into two 1D maps. The coupling is described by the pruning front or the symbolic representation of the partition line. Since an attractor is only related to backward foliations, which are the generalization of unstable manifolds, forward foliations were not considered in Ref. [11]. In order to determine a partition line more accurately we directly search for tangencies between forward and backward foliations. The way to determine the tangent direction of a point on a foliation is simple. Take the given point as an end point of an orbit. At some point on the orbit we pick up a rather arbitrary direction, which will map to some direction at the end point. If we fix the end point and increase the length of the orbit, due to the instability of the dynamics, the final direction will usually converge to a certain direction, which is the unstable or backward direction of the given point [6]. A backward foliation is an integral curve of the field of such directions. A forward direction and forward foliation can be determined similarly [7]. The technical details will be presented elsewhere [12]. For the parameters $\sigma=4.875$, $A=0.018$, $\omega=0.03168$, $a=0.2621$, $r=1.807$, and $b=0.0002$, as given in Ref. [11], the obtained partition line is shown in Fig. 1. The Poincaré section is constructed at $t=2\pi(n+77/256)/\omega$ in the $y$-$z$ plane.

It was claimed that the partition for the EBL model is

 

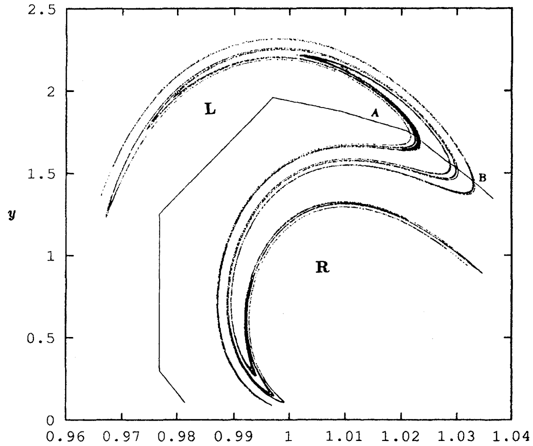

FIG. 1. The partition line ($AB$) in the $y$-$z$ plane determined from tangencies between forward and backward foliations divides the plane into two parts marked $R$ and $L$. Phase offset $77\pi/128\omega$ is taken for the Poincaré section. Here the variables $y$ and $z$ have been scaled with $y^* = \sqrt{b(r-1)}$ and $z^* = r-1$, respectively. A part of the attractor is also shown.

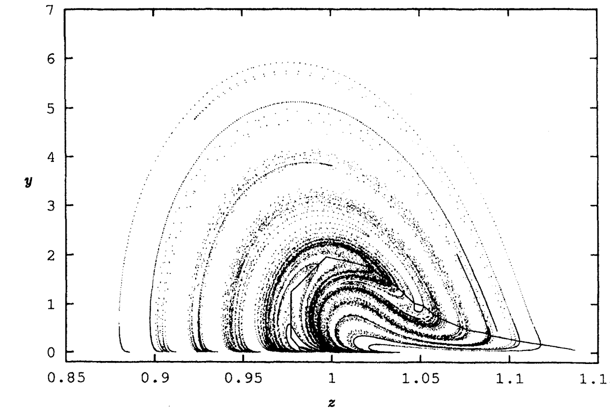

FIG. 2. The partition line constructed in a larger region of the phase space (cf. Fig. 1).

binary [11]. This is also verified by tangencies of foliations. The symbolic dynamics of the Hénon map is known to be binary. Poincaré maps of differential systems usually have a positive Jacobian. We find that the ordering rules of forward and backward sequences for the EBL model coincide with those for the Hénon map with a positive Jacobian. They are

$$\bullet ER \ldots > \bullet EL \ldots, \quad \bullet OR \ldots < \bullet OL \ldots ;$$
$$\ldots RE\bullet > \ldots LE\bullet, \quad \ldots RO\bullet < \ldots LO\bullet , \tag{2}$$

where finite strings $E$ and $O$ consist of letters $R$ and $L$, and contain an even and odd number of letter $R$, respectively. Relations (2) can be understood from the "local" ordering that both eigenvalues of the fixed point $R^\infty$ (and $L^\infty$) for the Hénon map with a positive Jacobian are negative (and positive). From the ordering rules the greatest backward sequence is $L^\infty R\bullet$. However, in Ref. [11] the greatest backward sequence seen for the given parameters was not greater than $(RLR)^\infty\bullet$. We then search a larger region (beyond $L^5R\bullet$) of the phase space for more tangencies and construct a more complete partition line, which is shown in Fig. 2.

Since foliations are well ordered, the geometry of a tangency places a restriction on allowed symbolic sequences. A point of tangency on the partition line $C\bullet$ may symbolically be represented as $QC\bullet P$. The sequence $UV$ where $U\bullet$ is between $QR\bullet$ and $QL\bullet$, and $\bullet V > \bullet P$ must be forbidden by the tangency $QC\bullet P$. (This is the meaning of a pruning or forbidden rectangle in the symbolic plane [5,13].) Consider a finite set of tangencies $\{Q_iC\bullet P_i\}$. If the shift of a sequence $\ldots s_{k-1}\bullet s_k s_{k+1}\ldots$ satisfies the condition that the backward sequence $\ldots s_{k-2}s_{k-1}\bullet$ is not between $Q_iR\bullet$ and $Q_iL\bullet$, and at the same time $\bullet P_i > \bullet s_k s_{k+1}\ldots$ for some $i$, then this shift is not forbidden by any tangencies due to the property of well ordering of foliations. Thus we may say that the shift is allowed according to the tangency. If all shifts of the sequence are allowed according to the set of tangencies, then the sequence is admissible.

We select the following finite tangencies:

1: $\ldots LRRRLLLLLLRRLLLLRC\bullet RRLLRLLRRLRLRRLRLR\ldots$

2: $\ldots LLLLLLRRLLLLRLLLRC\bullet RRRRRRLRRLRLRRLRRR\ldots$

3: $\ldots LLRRLLLLRLLRLRRLRC\bullet RLRRRLRRRLRLRLRRRL\ldots$

4: $\ldots RLRLRLRLRLRLRRRLRC\bullet RLRRRLRLRLRRRLRRRL\ldots$

5: $\ldots LLLLRRLLLLLRRLRRRC\bullet RLRRRRLRLRLRLRLRLR\ldots$

6: $\ldots RRRLRRLRLRRLRLRRRC\bullet RLRRRRLRRLRLRLRLRR\ldots$

7: $\ldots RRLLLLLRLLRRLRLRRC\bullet RLRRLRRLRLRRRRRLRR\ldots$

8: $\ldots RRRLLLLLLRRLLLLRLC\bullet RLLRLRRLRLRRLRRRRL\ldots$

We have examined 15 orbits up to period 9 found in Ref. [11]. They are indeed allowed by the set of tangencies. Among sequences up to order 9 smaller than $RLC$ in the so-called MSS (Metropolis, Stein, and Stein) $U$ sequence, four periods, three orbits of period 9, and one period 7 are forbidden. However, two orbits of period 8, $(RLRRLRRL)^\infty$ and $(RLRRLRRR)^\infty$, not in the abovementioned 15 orbits, are also allowed. Contrary to the conclusion made in Ref. [11], periods $(RLL)^\infty$, $(RLLRLR)^\infty$, $(RLLRLRRLL)^\infty$, and $(RLLRLRRRLR)^\infty$

are allowed. In fact, there are more allowed orbits, e.g., $(RLLR)^\infty$ and $(RLLL)^\infty$, but to verify their admissibility, more tangencies must be considered. For all the inferred allowed sequences, we have numerically found their orbits in phase space, using the ordering of foliations to choose initial values for the Newton method of iteration. So far we have mentioned only periodic orbits. We may further construct allowed chaotic sequences based on the set of tangencies. For example, it can be verified that any sequences consisting of only the segments $RLR^5L$ and $R^5L$ are always allowed. A more detailed characterization of the symbolic dynamics for the EBL model, including the symbolic analysis of crisis, will be presented elsewhere [12].

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